Chapter 7

Pondering Polynomials

In This Chapter

- Providing techniques for making graphing polynomials easier
- Segueing from intercepts to roots of polynomials
- Solving polynomial equations using everything but the kitchen sink

The word *polynomial* comes from *poly-*, meaning "many," and *-nomial,* meaning "name" or "designation." The exponents used in polynomials are all whole numbers — no fractions or negatives. Polynomials get progressively more interesting as the exponents get larger — they can have more intercepts and turning points. This chapter outlines how to deal with polynomials: factoring them, graphing them, analyzing them. The graph of a polynomial looks like a Wisconsin landscape — smooth, rolling curves. Are you ready for this ride?

Sizing Up a Polynomial Equation

A *polynomial function* is a specific type of function that can be easily spotted in a crowd of other types of functions and equations. By convention, you write the terms from the largest exponent to the smallest.

The general form for a polynomial function is

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_1 x^1 + a_0$$

Here, the *a*'s are real numbers and the *n*'s are whole numbers. The last term is technically $a_0 x^0$, if you want to show the variable in every term.

Identifying Intercepts and Turning Points

The *intercepts* of a polynomial are the points where the graph of the curve of the polynomial crosses the *x*-axis and *y*-axis. A polynomial function has *exactly* one *y*-intercept, but it can have many *x*-intercepts, depending on the degree of the polynomial (the highest power of the variable). The higher the degree, the more *x*-intercepts are possible.

The *x-intercepts* of a polynomial are also called the *roots, zeros,* or *solutions.* The *x*-intercepts are often where the graph of the polynomial goes from positive values (above the *x*-axis) to negative values (below the *x*-axis) or from negative values to positive values. Sometimes, though, the values on the graph don't change sign at an *x*-intercept: These graphs look sort of like a *touch and go.* The curves approach the *x*-axis, seem to change their minds about crossing the axis, touch down at the intercepts, and then go back to the same side of the axis.

A *turning point* of a polynomial is where the graph of the curve changes direction. It can change from going upward to going downward, or vice versa. A turning point is where you find a maximum value of the polynomial or a minimum value.

Interpreting relative value and absolute value

A parabola opening downward has an absolute maximum — you see no point on the curve that's higher than the maximum. In other words, no value of the function is greater than the function value at that point. Some functions, however, also have *relative* maximum or minimum values:

- **Relative maximum:** A function value that is bigger than all function values around it — it's *relatively* large. The function value is bigger than anything around it, but you may be able to find a bigger function value somewhere else.
- **Relative minimum:** A function value that is smaller than all function values around it. The function value is smaller than anything close to it, but there may be a function value that's smaller somewhere else.

In Figure 7-1, you can see five turning points. Two correspond to relative maximum values, which means they're higher than any points close to them. Three correspond to minimum values, which means they're lower than any points around them. Two of the minimums correspond to relative minimum values, and one has absolutely the lowest function value on the curve. This function has no absolute maximum value because it keeps going up and up without end.

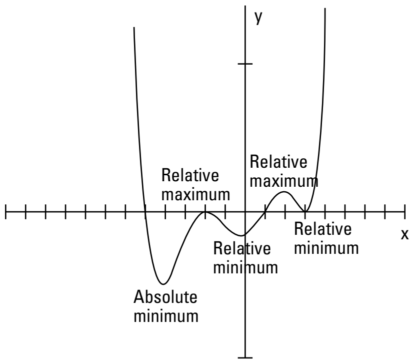


Figure 7-1: Extreme points on a polynomial.

Dealing with intercepts and turning points

The number of potential turning points and *x*-intercepts of a polynomial function is good to know when you're sketching the graph of the function. You can often count the number of *x*-intercepts and turning points of a polynomial if you have the graph of it in front of you, but you can also make an estimate of the number if you have the equation of the polynomial. Your estimate is actually a number that represents the most points that can occur.

ALGEBRA RULES

Given the polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x^1 + a_0$, the maximum number of *x*-intercepts is *n*, the degree or highest power of the polynomial. The maximum number of turning points is $n - 1$, or one less than the number of possible intercepts. You may find fewer *x*-intercepts than *n*, or you may find exactly that many.

EXAMPLE

Examine the function equations for intercepts and turning points:

$$f(x) = 2x^7 + 9x^6 - 75x^5 - 317x^4 + 705x^3 + 2{,}700x^2$$

This graph has at most seven x-intercepts (7 is the highest power in the function) and six turning points (7 – 1).

You can see the graph of the function in Figure 7-2. According to its equation, the graph of the polynomial could have as many as seven x-intercepts, but it has only five; it does have all six turning points, though. You can also see that two of the intercepts are touch-and-go types, meaning that they approach the x-axis before heading away again.

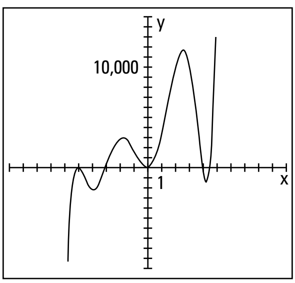


Figure 7-2: The intercept and turning-point behavior of a polynomial function.

Solving for y-intercepts and x-intercepts

You can easily solve for the y-intercept of a polynomial function; the y-intercept is where the curve of the graph crosses the y-axis, and that's when $x = 0$. So, to determine the y-intercept for any polynomial, simply replace all the x's with zeros and solve for y (that's the y part of the coordinates of that intercept). For example, in $y = 3x^4 - 2x^2 + 5x - 3$, you get $y = 3(0)^4 - 2(0)^2 + 5(0) - 3 = -3$, so the y-intercept is (0, –3).

After you complete the easy task of solving for the y-intercept, you find out that the x-intercepts are another matter altogether. The value of y is 0 for all x-intercepts, so you let $y = 0$ and solve.

When the polynomial is factorable, you use the multiplication property of zero (MPZ; see Chapter 1), setting the factored form equal to 0 to find the x-intercepts.

Determine the x-intercepts of the polynomial $y = x^3 - 16x$.

Replace the y with zeros and solve for x:

$$0 = x^3 - 16x = x(x^2 - 16) = x(x - 4)(x + 4)$$

Using the MPZ, you get that $x = 0$, $x = 4$, or $x = -4$. The x-intercepts are (0, 0), (4, 0), and (–4, 0).

Determining When a Polynomial Is Positive or Negative

When a polynomial has positive y-values for some interval — between two x-values — its graph lies above the x-axis in that interval. When a polynomial has negative values, its graph lies below the x-axis in that interval. The only way for a polynomial to change from positive to negative values or vice versa is to go through 0 — at an x-intercept.

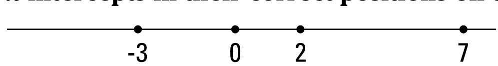
Incorporating a sign line

If you're a visual person like me, you'll appreciate the interval method I present in this section. Using a *sign line* and marking the intervals between x-values allows you to determine where a polynomial is positive or negative, and it appeals to your artistic bent!

Determine when the function $f(x) = x(x - 2)(x - 7)(x + 3)$ is positive and when it's negative.

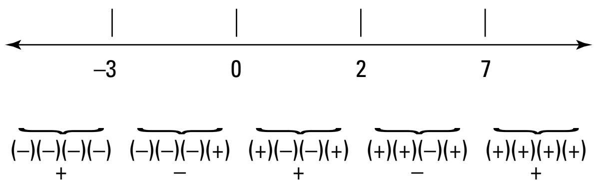
Setting $f(x) = 0$ and solving, you find that the x-intercepts are at $x = 0$, 2, 7, and –3. To determine the positive and negative intervals for a polynomial function, follow this method:

1. **Draw a number line, and place the values of the x-intercepts in their correct positions on the line.**



2. Choose random values to the right of and left of and in between the intercepts to test whether the function is positive or negative in those intervals.

One efficient method is to insert the “test values” into the factored form of the polynomial and just record the signs — which then give you the positive or negative result for the entire interval.



You need to check only one point in each interval; the function values all have the same sign within that interval.

The graph of this function is positive, or above the *x*-axis, whenever *x* is smaller than –3, between 0 and 2, or bigger than 7. You write this part of the answer as: $x < -3$ or $0 < x < 2$ or $x > 7$. The graph of the function is negative when $-3 < x < 0$ or $2 < x < 7$.

Recognizing a sign change rule

In the previous example, you see the signs changing at each intercept. If the signs of functions don’t change at an intercept, then the graph of the polynomial doesn’t cross the *x*-axis at that intercept, and you see a touch-and-go. It’s nice to be able to predict such behavior.

The rule for whether a function displays sign changes or not at the intercepts is based on the exponent on the factor that provides you with a particular intercept.

If a polynomial function is factored in the form $y = (x - a_1)^{n_1}(x - a_2)^{n_2} \ldots$, you see a sign change at a_1 whenever n_1 is an odd number (meaning it crosses the *x*-axis), and you see no sign change whenever n_1 is even (meaning the graph of the function is touch-and-go; see the “Dealing with intercepts and turning points” section, earlier in this chapter).

So, for example, with the function $y = x^4(x - 3)^3(x + 2)^8(x + 5)^2$, you'll find a sign change at $x = 3$ and no sign change at $x = 0$, -2, or -5. And with the function $y = (2 - x)^2(4 - x)^2(6 - x)^2(2 + x)^2$, you never see a sign change — the function is always either positive or just touching the *x*-axis.

Solving Polynomial Equations

Finding intercepts (or roots or zeros) of polynomials can be relatively easy or a little challenging, depending on the complexity of the function. Polynomials that factor easily are very desirable. Polynomials that don't factor at all, however, are relegated to computers or graphing calculators.

The polynomials that remain are those that factor — but take a little planning and work. The planning process involves counting the number of possible positive and negative real roots and making a list of potential rational roots. The work is done using synthetic division to test the list of choices to find the roots.

Factoring for roots

Finding *x*-intercepts of polynomials isn't difficult — as long as you have the polynomial in nicely factored form. You just set the *y* equal to 0 and use the MPZ. This section deals with easily recognizable factors of polynomials; I cover other, more challenging types in the following sections.

Half the battle when factoring is recognizing the patterns in factorable polynomial functions. Here are the most easily recognizable factoring patterns used on polynomials:

- **Difference of squares:** $a^2 - b^2 = (a + b)(a - b)$.
- **Greatest common factor (GCF):** $ab \pm ac = a(b \pm c)$.
- **Difference of cubes:** $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.
- **Sum of cubes:** $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.
- **Perfect square trinomial:** $a^2 \pm 2ab + b^2 = (a \pm b)^2$.
- **Trinomial factorization:** UnFOIL (see Chapter 1).
- **Common factors in groups:** Grouping (see Chapter 1).

The following examples incorporate the different methods of factoring. They contain perfect cubes and squares and all sorts of good combinations of factorization patterns.

Factor the polynomial: $y = 4x^5 - 25x^3$.

First use the GCF and then the difference of squares:

$$y = 4x^5 - 25x^3 = x^3(4x^2 - 25) = x^3(2x - 5)(2x + 5)$$

Factor the polynomial: $y = 64x^8 - 64x^6 - x^2 + 1$.

You initially factor the polynomial by grouping. The first two terms have a common factor of $64x^6$, and the second two terms have a common factor of –1. The new equation has a common factor of $x^2 - 1$. After performing the factorization, you see that both factors are the difference of squares:

$$\begin{aligned} y &= 64x^8 - 64x^6 - x^2 + 1 \\ &= 64x^6\left(x^2 - 1\right) - 1\left(x^2 - 1\right) \\ &= \left(x^2 - 1\right)\left(64x^6 - 1\right) \end{aligned}$$

Now you factor the binomials as the difference of perfect squares. Then you can factor the last two new binomials using the difference and sum of two perfect cubes:

$$\begin{aligned} &= (x - 1)(x + 1)\left(8x^3 - 1\right)\left(8x^3 + 1\right) \\ &= (x - 1)(x + 1)(2x - 1)\left(4x^2 + 2x + 1\right)(2x + 1)\left(4x^2 - 2x + 1\right) \end{aligned}$$

The two trinomials resulting from factoring the difference and sum of cubes don't factor, so you're done. Whew!

Taking sane steps with the rational root theorem

What do you do if the factorization of a polynomial doesn't leap out at you? You have a feeling that the polynomial factors, but the necessary numbers escape you. Never fear! The rational root theorem is here.

The *rational root theorem* states that if the polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x^1 + a_0$ has any rational roots, they all meet the requirement that you can write them as a fraction equal to $\frac{\text{factor of } a_0}{\text{factor of } a_n}$.

In other words, according to the theorem, any rational root of a polynomial with integer coefficients is formed by dividing a factor of the constant term by a factor of the lead coefficient. Of course, this means that the a_0 term, the constant, cannot be 0.

Taking the first step

The rational root theorem creates a list of numbers that may be roots of a particular polynomial. After using the theorem to make your list of potential roots, you plug the numbers into the polynomial to determine which, if any, work. You may run across an instance where none of the candidates work, which tells you that there are no rational roots. (And if a given rational number isn't on the list of possibilities that you come up with, it can't be a root of that polynomial.)

Before you start to plug and chug, however, check out the "Putting Descartes in charge of signs" section, later in this chapter — it helps you with your guesses. Also, you can refer to "Finding Roots Synthetically," later in this chapter, for a quicker method than plugging in.

To find the rational roots of the polynomial $y = x^4 - 3x^3 + 2x^2 + 12$, for example, you test the following possibilities: ±1, ±2, ±3, ±4, ±6, and ±12. These values are all the factors of the number 12. Technically, you divide each of these factors of 12 by the factors of the lead coefficient, but because the lead coefficient is one (as in $1x^4$), dividing by that number won't change a thing.

Find the roots of the polynomial $y = 6x^7 - 4x^4 - 4x^3 + 2x - 20$.

You first list all the factors of 20: ±1, ±2, ±4, ±5, ±10, and ±20. Now divide each of those factors by the factors of 6. You don't need to bother dividing by 1 to create your list, but you need to divide each by 2, 3, and 6: $\pm\frac{1}{2}, \pm\frac{2}{2}, \pm\frac{4}{2}, \pm\frac{5}{2}, \pm\frac{10}{2}, \pm\frac{20}{2}, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}, \pm\frac{5}{3}, \pm\frac{10}{3}, \pm\frac{20}{3}, \pm\frac{1}{6}, \pm\frac{2}{6}, \pm\frac{4}{6}, \pm\frac{5}{6}, \pm\frac{10}{6}, \pm\frac{20}{6}$. And, of course, you include : ±1, ±2, ±4, ±5, ±10, and ±20 as candidates.

You may have noticed some repeats in the previous list that occur when you reduce fractions. You can discard the repeats. And, even though this looks like a mighty long list, between the integers and fractions, it still gives you a reasonable number of candidates to try out. You can check them off in a systematic manner.

Changing from roots to factors

When you have the factored form of a polynomial and set it equal to 0, you can solve for the solutions (or x-intercepts, if that's what you want). Just as important, if you have the solutions, you can go backward and write the factored form. Factored forms are needed when you have polynomials in the numerator and denominator of fractions and you want to reduce the fraction. Factored forms are easier to compare with one another.

How can you use the rational root theorem to factor a polynomial function? Why would you want to? The answer to the second question, first, is that you can reduce a factored form if it's in a fraction. Also, a factored form is more easily graphed. Now, for the first question: You use the rational root theorem to find roots of a polynomial and then translate those roots into binomial factors whose product is the polynomial.

If $x = \frac{b}{a}$ is a root of the polynomial $f(x)$, the corresponding binomial $(ax - b)$ is a factor.

Write the factorization of a polynomial with the five roots $x = 1$, $x = -2$, $x = 3$, $x = \frac{3}{2}$, and $x = -\frac{1}{2}$.

Applying the rule, you get $f(x) = (x - 1)(x + 2)(x - 3)(2x - 3)(2x + 1)$. Notice that the positive roots give factors of the form $x - c$, and the negative roots give factors of the form $x + c$, which comes from $x - (-c)$. This is just one polynomial with these five roots. You can write other polynomials by multiplying the factorization by some constant.

To show *multiple roots*, or roots that occur more than once, use exponents on the factors. For example, if the roots of a polynomial are $x = 0$, $x = 2$, $x = 2$, $x = -3$, $x = -3$, $x = -3$, $x = -3$, and $x = 4$, a corresponding polynomial is $f(x) = x(x - 2)^2(x + 3)^4(x - 4)$.

Putting Descartes in charge of signs

Descartes' rule of signs tells you how many positive and negative *real* roots you may find in a polynomial. A *real number* is just about any number you can think of. It can be positive or negative, rational or irrational. The only thing it can't be is imaginary.

Counting up the number of possible positive roots

The first part of the rule of signs helps you identify how many of the roots of a polynomial are positive.

Descartes' rule of signs (part I): The polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x^1 + a_0$ has at most n roots. Count the number of times the sign changes in the coefficients of f, and call that value p. The value of p is the maximum number of *positive* real roots of f. If the number of positive roots isn't p, it is $p - 2$, $p - 4$, or some number less by a multiple of 2.

Use part I of Descartes' rule of signs on the polynomial $f(x) = 2x^7 - 19x^6 + 66x^5 - 95x^4 + 22x^3 + 87x^2 - 90x + 27$.

Count the number of sign changes. The sign of the first term starts as a positive, changes to a negative, and moves to positive; negative; positive; stays positive; negative; and then positive. Whew! In total, you count six sign changes. Therefore, you conclude that the polynomial has six positive roots, four positive roots, two positive roots, or none at all. When a root, such as $x = 3$ in the previous example, occurs more than once, you say that the root has *multiplicity* two or three or however many times it appears. This way, if you count the root as many times as it appears, the total will correspond to your predicted number.

Counting the possible number of negative roots

Along with the positive roots (see the previous section), Descartes' rule of signs deals with the possible number of negative roots of a polynomial. After you count the possible number of positive roots, you combine that value with the number of possible negative roots to make your guesses and solve the equation.

Descartes' rule of signs (part II): The polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x^1 + a_0$ has at most n roots. Find $f(-x)$, and then count the number of times the sign changes in $f(-x)$ and call that value q. The value of q is the maximum number of *negative* roots of f. If the number of negative roots isn't q, the number is $q - 2$, $q - 4$, and so on, for as many multiples of 2 as necessary. Again, you count a multiple root as many times as it occurs when applying the rule.

Determine the possible number of negative roots of the polynomial $f(x) = 2x^7 - 19x^6 + 66x^5 - 95x^4 + 22x^3 + 87x^2 - 90x + 27$.

You first find $f(-x)$ by replacing each x with $-x$ and simplifying:

$$f(-x) = 2(-x)^7 - 19(-x)^6 + 66(-x)^5 - 95(-x)^4 + 22(-x)^3 + 87(-x)^2 - 90(-x) + 27 = -2x^7 - 19x^6 - 66x^5 - 95x^4 - 22x^3 + 87x^2 + 90x + 27$$

As you can see, the function has only one sign change, from negative to positive. Therefore, the function has exactly one negative root — no more, no less. In fact, this negative root is -1.

Knowing the potential number of positive and negative roots for a polynomial is very helpful when you want to pinpoint an exact number of roots. The example polynomial I present in this section has only one negative real root. That fact tells you to concentrate your guesses on positive roots; the odds are better that you'll find a positive root first.

Finding Roots Synthetically

You use synthetic division to test the list of possible roots for a polynomial that you come up with by using the rational root theorem. *Synthetic division* is a method of dividing a polynomial by a binomial, using only the coefficients of the terms. The method is quick, neat, and highly accurate — usually even more accurate than long division, because it has fewer opportunities for "user error."

Using synthetic division when searching for roots

When you use synthetic division to look for roots in a polynomial, the last number on the bottom row of your synthetic division problem is the telling result. If that number is 0, the division had no remainder, and the number is a root. The fact that there's no remainder means that the binomial represented by the number is dividing the polynomial evenly. The number is a root because the binomial is a factor of the polynomial.

Use synthetic division, the rational root theorem, and Descartes' rule of signs to find roots of the polynomial $f(x) = x^5 + 5x^4 - 2x^3 - 28x^2 - 8x + 32$.

Using the rational root theorem, your list of the potential rational roots is ±1, ±2, ±4, ±8, ±16, and ±32.

Then, applying Descartes' rule of signs, you determine that there are two or zero positive real roots and three or one negative real roots.

Here are the steps for performing synthetic division on a polynomial to find its roots:

1. **Write the polynomial in order of decreasing powers of the exponents. Replace any missing powers with 0 to represent the coefficient.**

   In this case, you've lucked out. The polynomial is already in the correct order: $f(x) = x^5 + 5x^4 - 2x^3 - 28x^2 - 8x + 32$.

2. **Write the coefficients in a row, including the zeros.**

   1 5 –2 –28 –8 32

3. **Put the number you want to divide by in front of the row of coefficients, separated by a half-box. Then draw a horizontal line below the row of coefficients, leaving room for numbers under the coefficients.**

   In this case, my guess is $x = 1$.

   1| 1 5 –2 –28 –8 32

   ____

4. **Bring the first coefficient straight down below the line. Then multiply the number you bring below the line by the number that you're dividing into everything. Put the result under the second coefficient.**

$$\begin{array}{r|rrrrrr} 1 & 1 & 5 & -2 & -28 & -8 & 32 \\ & & 1 & & & & \\ \hline & 1 & & & & & \end{array}$$

5. **Add the second coefficient and the product, putting the result below the line.**

$$\begin{array}{r|rrrrrr} 1 & 1 & 5 & -2 & -28 & -8 & 32 \\ & & 1 & & & & \\ \hline & 1 & 6 & & & & \end{array}$$

6. **Repeat the multiplication/addition with the rest of the coefficients.**

$$\begin{array}{r|rrrrrr} 1 & 1 & 5 & -2 & -28 & -8 & 32 \\ & & 1 & 6 & 4 & -24 & -32 \\ \hline & 1 & 6 & 4 & -24 & -32 & 0 \end{array}$$

The last entry on the bottom is a 0, so you know 1 is a root. Now, you can do a modified synthetic division when testing for the next root; you just use the numbers across the bottom. (These values are actually coefficients of the quotient, if you do long division; see the following section.)

If your next guess is to see if $x = -1$ is a root, the modified synthetic division appears as follows:

$$\begin{array}{r|rrrrr} -1 & 1 & 6 & 4 & -24 & -32 \\ & & -1 & -5 & 1 & 23 \\ \hline & 1 & 5 & -1 & -23 & -9 \end{array}$$

The last entry on the bottom row isn't 0, so –1 isn't a root.

The really good guessers amongst you decide to try $x = 2$, $x = -4$, $x = -2$, and $x = -2$ (a second time). These values represent the rest of the roots.

Synthetically dividing by a binomial

Finding the roots of a polynomial isn't the only excuse you need to use synthetic division. You can also use synthetic division to replace the long, drawn-out process of dividing a polynomial by a binomial. The polynomial can be any degree; the binomial has to be either $x + c$ or $x - c$, and the coefficient on the x is 1. This may seem rather restrictive, but a huge number of long divisions you'd have to perform fit in this category, so it helps to have a quick, efficient method to perform these basic division problems.

To use synthetic division to divide a polynomial by a binomial, you first write the polynomial in decreasing order of exponents, inserting a 0 for any missing exponent. The number you put in front or divide by is the *opposite* of the number in the binomial.

Divide $2x^5 + 3x^4 - 8x^2 - 5x + 2$ by the binomial $x + 2$ using synthetic division.

Using –2 in the synthetic division:

–2	2	3	0	–8	–5	2
		–4	2	–4	24	–38
	2	–1	2	–12	19	–36

As you can see, the last entry on the bottom row isn't 0. If you're looking for roots of a polynomial equation, this fact tells you that –2 isn't a root. In this case, because you're working on a long division application, the –36 is the remainder of the division — in other words, the division doesn't come out even.

You obtain the answer (quotient) of the division problem from the coefficients across the bottom of the synthetic division. You start with a power one value lower than the original polynomial's power, and you use all the coefficients, dropping the power by one with each successive coefficient. The last coefficient is the remainder, which you write over the divisor.

Here's the division problem and its solution. The original division problem is written first. Under the problem, you see the coefficients from the synthetic division written in front of variables — starting with one degree lower than the original problem. The remainder of –36 is written in a fraction on top of the divisor, $x + 2$.

$$\left(2x^5 + 3x^4 - 8x^2 - 5x + 2\right) \div \left(x + 2\right) =$$
$$2x^4 - x^3 + 2x^2 - 12x + 19 - \frac{36}{x + 2}$$